POLS 8505 Measurement Theory

12 January 2015

1. Notation:

   a. $\mathbf{x_{ik}}$ is the ith individual's coordinate on the kth dimension $\mathbf{i=1,\ldots,p}$, where $\mathbf{p}$ is the number of individuals; and $\mathbf{k=1,\ldots,s}$, where $\mathbf{s}$ is the number of dimensions. Represented as a vector:

$$\underline{\mathbf{x}}_i = \begin{bmatrix} \mathbf{x}_{i1} \\ \mathbf{x}_{i2} \\ \mathbf{x}_{i3} \\ \vdots \\ \mathbf{x}_{is} \end{bmatrix}$$

   b. $\mathbf{z_{jk}}$ is the jth stimulus coordinate on the kth dimension $\mathbf{j=1,\ldots,q}$, where $\mathbf{q}$ is the number of stimuli. Represented as a vector:

$$\underline{\mathbf{z}}_j = \begin{bmatrix} \mathbf{z}_{j1} \\ \mathbf{z}_{j2} \\ \mathbf{z}_{j3} \\ \vdots \\ \mathbf{z}_{js} \end{bmatrix}$$

   c. Simple Euclidean distance between the ith individual and the jth stimulus:

$$\mathbf{d}(\underline{\mathbf{x}}_i, \underline{\mathbf{z}}_j) = \mathbf{d}_{ij} = \|\underline{\mathbf{x}}_i - \underline{\mathbf{z}}_j\| = \sqrt{\sum_{k=1}^{s} (\mathbf{x}_{ik} - \mathbf{z}_{jk})^2}$$

   d. Properties of distances:

$$\mathbf{d(\underline{x}_i,\ \underline{x}_i) = 0}$$
$$\mathbf{d(\underline{x}_i,\ \underline{z}_j) > 0 \text{ if } \underline{x}_i \neq \underline{z}_j}$$
$$\mathbf{d(\underline{x}_i,\ \underline{z}_j) = d(\underline{z}_j,\ \underline{x}_i)}$$
$$\mathbf{d(\underline{x}_i,\ \underline{z}_j) \leq d(\underline{x}_i,\ \underline{y}_h) + d(\underline{y}_h,\ \underline{z}_j)}$$

e. Minkowski p-metric

$$\mathbf{d_{ij}(p) = \left[\sum_{k=1}^{s} | x_{ik} - z_{jk} |^{p}\right]^{\frac{1}{p}},\ \ p \geq 1}$$

f. Matrix Notation

$$\mathbf{X} = \begin{bmatrix} \mathbf{x_{11}} & \mathbf{x_{12}} & . & . & . & \mathbf{x_{1s}} \\ \mathbf{x_{21}} & \mathbf{x_{22}} & . & . & . & \mathbf{x_{2s}} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \mathbf{x_{p1}} & \mathbf{x_{p2}} & . & . & . & \mathbf{x_{ps}} \end{bmatrix}$$

$$\mathbf{Z} = \begin{bmatrix} \mathbf{z_{11}} & \mathbf{z_{12}} & . & . & . & \mathbf{z_{1s}} \\ \mathbf{z_{21}} & \mathbf{z_{22}} & . & . & . & \mathbf{z_{2s}} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \mathbf{z_{q1}} & \mathbf{z_{q2}} & . & . & . & \mathbf{z_{qs}} \end{bmatrix}$$

The q by q ***Similarities Matrix*** (technically, the ***Dis-similarities Matrix*** of squared distances for the Stimuli is:

$$
\mathbf{D_z} = \begin{bmatrix}
\sum_{k=1}^{s}(z_{1k} - z_{1k})^2 & \sum_{k=1}^{s}(z_{1k} - z_{2k})^2 & . \quad . \quad . & \sum_{k=1}^{s}(z_{1k} - z_{qk})^2 \\
\sum_{k=1}^{s}(z_{2k} - z_{1k})^2 & \sum_{k=1}^{s}(z_{2k} - z_{2k})^2 & . \quad . \quad . & \sum_{k=1}^{s}(z_{2k} - z_{qk})^2 \\
. & . & . \qquad\quad & . \\
. & . & \quad . \quad & . \\
. & . & \qquad\quad . & . \\
\sum_{k=1}^{s}(z_{qk} - z_{1k})^2 & \sum_{k=1}^{s}(z_{qk} - z_{2k})^2 & . \quad . \quad . & \sum_{k=1}^{s}(z_{qk} - z_{qk})^2
\end{bmatrix}
$$

The equivalent expression for the p by q matrix of squared distances between **X** and **Z** (individuals and stimuli – *the **unfolding problem***) is:

$$
\mathbf{D} = \begin{bmatrix}
\sum_{k=1}^{s}(x_{1k} - z_{1k})^2 & \sum_{k=1}^{s}(x_{1k} - z_{2k})^2 & . \quad . \quad . & \sum_{k=1}^{s}(x_{1k} - z_{qk})^2 \\
\sum_{k=1}^{s}(x_{2k} - z_{1k})^2 & \sum_{k=1}^{s}(x_{2k} - z_{2k})^2 & . \quad . \quad . & \sum_{k=1}^{s}(x_{2k} - z_{qk})^2 \\
. & . & . \qquad\quad & . \\
. & . & \quad . \quad & . \\
. & . & \qquad\quad . & . \\
\sum_{k=1}^{s}(x_{pk} - z_{1k})^2 & \sum_{k=1}^{s}(x_{pk} - z_{2k})^2 & . \quad . \quad . & \sum_{k=1}^{s}(x_{pk} - z_{qk})^2
\end{bmatrix}
$$