CONJUGATE PRIOR DISTRIBUTIONS

1. Sampling From a Bernoulli Distribution: The joint distribution of the sample ("Likelihood Function") is proportional to a Binomial Distribution. When multiplied by a Beta Prior Distribution it yields a Beta Posterior Distribution:

$$f_n(\underline{x} \mid p) = \prod_{i=1}^{n} f(x_1 \mid p) f(x_2 \mid p) \ldots f(x_n \mid p) = p^{\sum_{i=1}^{n} X_i} (1 - p)^{n - \sum_{i=1}^{n} X_i}$$

$$\xi(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} \propto p^{\alpha-1}(1-p)^{\beta-1}$$

$$\xi(p \mid \underline{x}) = \propto f_n(\underline{x} \mid p)\xi(p) \propto p^{\alpha+y-1}(1-p)^{\beta+n-y-1} \quad \textbf{where} \quad y = \sum_{i=1}^{n} X_i$$

2. Sampling From a Poisson Distribution: The joint distribution of the sample ("Likelihood Function") multiplied by a Gamma Prior Distribution yield a Gamma Posterior Distribution:

$$f_n(\underline{x} \mid \lambda) = \prod_{i=1}^{n} f(x_1 \mid \lambda) f(x_2 \mid \lambda) \ldots f(x_n \mid \lambda) \propto e^{-n\lambda} \lambda^{\sum_{i=1}^{n} X_i}$$

$$\xi(\lambda) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \propto \lambda^{\alpha-1} e^{-\beta\lambda}$$

$$\xi(\lambda \mid \underline{x}) = \propto f_n(\underline{x} \mid \lambda)\xi(\lambda) \propto \lambda^{\alpha+y-1} e^{(\beta+n)\lambda} \quad \textbf{where} \quad y = \sum_{i=1}^{n} X_i$$

3. Sampling From a Normal Distribution: Normal Joint Distribution of the Sample ("Likelihood Function") and

Normal Prior Distribution yield a Normal Posterior Distribution:

$$f_n(\underline{x} \mid \mu) = \prod_{i=1}^{n} f(x_1|\mu,\sigma^2)f(x_2|\mu,\sigma^2)...f(x_n|\mu,\sigma^2) \propto e^{-\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{2\sigma^2}} \propto e^{-\frac{n}{2\sigma^2}(\mu-\bar{X}_n)^2}$$

Using the fact that $\sum_{i=1}^{n}(X_i-\mu)^2 = n(\mu-\bar{X}_n)^2 + \sum_{i=1}^{n}(X_i-\bar{X}_n)^2$

Let the prior be a Normal with mean **π** and variance **ν²**:

$$\xi(\mu) \propto e^{-\frac{1}{2\nu^2}(\mu-\pi)^2}$$

$$\xi(\mu \mid \underline{x}) = \propto f_n(\underline{x} \mid \mu)\xi(\mu) \propto e^{-\frac{1}{2}\left[\frac{n}{\sigma^2}(\mu-\bar{X}_n)^2+\frac{1}{\nu^2}(\mu-\pi)^2\right]} \propto e^{-\frac{1}{2\nu_1^2}(\mu-\mu_1)^2}$$

Which is proportional to a Normal distribution with mean μ_1 and variance ν_1^2 where:

$$\mu_1 = \frac{\sigma^2}{\sigma^2+n\nu^2}\pi + \frac{n\nu^2}{\sigma^2+n\nu^2}\bar{X}_n \quad \textbf{and} \quad \nu_1^2 = \frac{\sigma^2\nu^2}{\sigma^2+n\nu^2}$$

4. Sampling From an Exponential Distribution: Exponential Joint Distribution of the Sample ("Likelihood Function") times a Gamma Prior Distribution yields a Gamma Posterior Distribution:

$$f_n(\underline{x} \mid \theta) = \prod_{i=1}^{n} f(x_1| \theta)f(x_2| \theta)...f(x_n| \theta) \propto \theta^n e^{-\theta\sum_{i=1}^{n}X_i}$$

$$\xi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta} \propto \theta^{\alpha-1}e^{-\beta\theta}$$

$$\xi(\theta \mid \underline{x}) = \propto f_n(\underline{x} \mid \theta)\xi(\theta) \propto \theta^{\alpha+n-1}e^{-(\beta+y)\theta}$$ **where** $y = \sum_{i=1}^{n} X_i$