The Relationship Between the Aldrich-McKelvey Scaling Solution and and the Individual Differences Problem

Keith T. Poole
University of Houston

In the individual differences problem, the individuals report judged similarities/dissimilarities between objects/stimuli. Let $\mathbf{d}^{*2}_{\mathbf{ijm}}$ be the reported dissimilarity by individual **i** (**i=1,...,p**) between stimulus **j** and stimulus **m** (**j,m=1,...,q**) and let $\mathbf{z}_{jk}$ be the true location of the **j**th stimulus on the **k**th dimension (**k=1,...,s**). The reported dissimilarity is assumed to be the sum of the true dissimilarity and a noise term:

$$\mathbf{d}^{*2}_{\mathbf{ijm}} = \sum_{\mathbf{k=1}}^{\mathbf{s}} \mathbf{w}^{2}_{\mathbf{ik}} (\mathbf{z}_{\mathbf{jk}} - \mathbf{z}_{\mathbf{jm}})^{2} + \mathbf{u}_{\mathbf{ijm}} \qquad \textbf{(1)}$$

where $\mathbf{w}^{2}_{\mathbf{ik}}$ is the weight individual **i** places on dimension **k** (hence the term "individual differences").

In the classic Carroll-Chang (1970) solution for the individual differences problem, the **p q** by **q** $\mathbf{D}^{*}$ matrices of dissimilarites (taken as squared distances) are double-centered and an alternating least squares procedure is used to estimate the **q** by **s** matrix **Z** and the **p q** by **q** diagonal matrices $\mathbf{W_i}$.

The purpose of this paper is to show an alternative solution for the individual differences problem using a method developed by Aldrich and McKelvey (1977) for scaling individuals' reported perceptions of the locations of stimuli along a scale with labeled endpoints.

In particular, let $\mathbf{z_{ji}}$ be the perceived location of stimulus **j** by individual **i**. Aldrich and McKelvey assume that the individual reports a noisy linear transformation of the true location of the stimulus; that is

$$\alpha_i + \beta_i \mathbf{z_{ji}} = \mathbf{z_j} + \mathbf{u_{ij}} \qquad (2)$$

where $\mathbf{u_{ij}}$ satisfies the usual Gauss-Markov assumptions of zero mean, constant variance across stimuli and individuals, and zero covariance.

Let $\hat{\mathbf{z}}_j$ be the estimated location of stimulus j and let $\hat{\alpha}_i$ and $\hat{\boldsymbol{\beta}}_i$ be the estimates of α and $\boldsymbol{\beta}$**.** Define

$$\mathbf{e_i} = \hat{\boldsymbol{\alpha}}_i + \hat{\boldsymbol{\beta}}_i \mathbf{z_{ji}} - \hat{\mathbf{z}}_j \qquad (3)$$

Aldrich and McKelvey set up the following Lagrangean multiplier problem:

$$\mathbf{L(\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, z_j, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = \sum_{i=1}^{p} e_i^2 + 2\lambda_1 \sum_{j=1}^{q} \hat{z}_j + \lambda_2 \left[\sum_{j=1}^{q} \hat{z}_j^2 - 1 \right]} \qquad (4)$$

that is, minimize the sum of squared error subject to the constraints that the estimated stimuli coordinates have zero mean and sum of squares equal to one. Define the **q** by **2** matrix **X** as

$$\mathbf{X_i} = \begin{bmatrix} \mathbf{1} & \mathbf{z_{1i}} \\ \mathbf{1} & \mathbf{z_{2i}} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathbf{1} & \mathbf{z_{qi}} \end{bmatrix}$$

then the solution for the individual transformations is simply the "least-squares regression of the reported on the actual (unknown) positions of the candidates" (Aldrich and McKelvey, p. 115). That is,

$$\begin{bmatrix} \hat{\boldsymbol{\alpha}}_i \\ \hat{\boldsymbol{\beta}}_i \end{bmatrix} = \left[\mathbf{X}_i'\mathbf{X}_i\right]^{-1} \mathbf{X}_i'\underline{\hat{\mathbf{z}}} \qquad (5)$$

where $\underline{\hat{\mathbf{z}}}$ is the **q** by **1** vector of the actual (unknown) positions of the candidates:

$$\underline{\hat{\mathbf{z}}} = \begin{bmatrix} \hat{\mathbf{z}}_1 \\ \hat{\mathbf{z}}_2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{\hat{\mathbf{z}}}_q \end{bmatrix}$$

To get the solution for $\underline{\hat{\mathbf{z}}}$, define the **q** by **q** matrix **A** as:

$$\mathbf{A} = \left[\sum_{i=1}^{p} \mathbf{X}_i (\mathbf{X}_i' \mathbf{X}_i)^{-1} \mathbf{X}_i' \right]$$

Aldrich and McKelvey show that the partial derivatives for the $\hat{\mathbf{z}}_j$ can be rearranged into the linear system:

$$\left[\mathbf{A} - \mathbf{pI}_q \right] \underline{\hat{\mathbf{z}}} = \lambda_2 \underline{\hat{\mathbf{z}}} \qquad (6)$$

where $\mathbf{I}_q$ is the **q** by **q** identity matrix. By equation (6), $\underline{\hat{\mathbf{z}}}$ is simply an eigenvector of the matrix $(\mathbf{A} - \mathbf{pI_q})$ and λ_2 is the corresponding eigenvalue.

To determine which of the q possible eigenvectors is the solution, Aldrich and McKelvey show that

$$-\lambda_2 = \sum_{i=1}^{p} \mathbf{e}_i^2 = -\underline{\hat{\mathbf{z}}} \left[\mathbf{A} - \mathbf{pI}_q \right] \underline{\hat{\mathbf{z}}} \qquad (7)$$

Hence, the solution is the eigenvector of $(\mathbf{A} - \mathbf{pI_q})$ "with the highest (negative) nonzero" eigenvalue. The solution for $\underline{\hat{\mathbf{z}}}$ from (6) can be taken back to equation (5) to solve for the individual transformation parameters.

Aldrich and McKelvey's ingenious solution is also a solution for the one dimensional individual differences problem. To see this, regard the reported dissimilarites from the individuals as responses on a scale labeled "zero dissimilarity" at one end and "very large dissimilarity" at the other end. The $\mathbf{d}^{*2}_{\mathbf{ijm}}$ (or $\mathbf{d}^{*}_{\mathbf{ijm}}$) from equation (1) play the role of $\mathbf{z_{ji}}$ in equation (2). In this framework, let **q*** be the number of stimuli. Then there are **q = q*(q*-1)/2** $\mathbf{z_{ji}}$'s for each individual. The **q*(q*-1)/2** unique elements of the dissimilariites matrix gathered from each individual are placed in $\mathbf{X_i}$. Therefore, $\underline{\hat{\mathbf{z}}}$ from equation (6) is of length **q*(q*-1)/2** and is a noisy linear transformation of the true dissimilarities; that is

$$\mathbf{a} + \mathbf{b}\underline{\hat{\mathbf{z}}} = \underline{\Delta} + \underline{\mathbf{u}} \qquad (8)$$

where **a** and **b** are scalars, $\underline{\Delta}$ is a **q*(q*-1)/2** length vector of the true dissimilarities, and $\underline{\mathbf{u}}$ is a **q*(q*-1)/2** length vector of error terms. Note that $\underline{\hat{\mathbf{z}}}$ *will not necessarily satisfy the triangle inequality* and that it does not matter whether or not the data gathered from the individuals is treated as distances or squared distances. To solve for $\underline{\Delta}$, simply take the estimated $\hat{\alpha}_i$ and $\hat{\boldsymbol{\beta}}_i$, solve for the "true" dissimilarities for each individual, and take the mean over the individuals as the estimator for $\underline{\Delta}$; that is:

$$\hat{\mathbf{d}}_{\mathbf{jm}} = \frac{\left[\sum_{\mathbf{i=1}}^{\mathbf{p}} \frac{\left(\hat{\mathbf{z}}_{\mathbf{jm}} - \hat{\boldsymbol{\alpha}}_{\mathbf{i}}\right)}{\hat{\boldsymbol{\beta}}_{\mathbf{i}}}\right]}{\mathbf{p}} \qquad (9)$$

Again, note that if the original data is regarded as squared distances, then equation (9) would be the formula for $\hat{\mathbf{d}}^2_{\mathbf{jm}}$. In any event, it is a simple matter to double-center the matrix of $\hat{\mathbf{d}}^2_{\mathbf{jm}}$ and obtain the estimate of the stimulus coordinates.

A nice feature of this approach is that it also generalizes to the s-dimensional case in which the individuals are constrained to have the same *ratio of weights* over the dimensions. That is, for individuals **i** and **h:** $\frac{\mathbf{w_{i1}}}{\mathbf{w_{i2}}} = \frac{\mathbf{w_{h1}}}{\mathbf{w_{h2}}}$ and so on.

References

Aldrich, John H. and Richard D. McKelvey. 1977. "A Method of Scaling with Applications to the 1968 and 1972 Presidential Elections. *American Political Science Review*, 71:111-130.

Carroll, J. Douglas and Jih-Jie Chang. 1970. "Analysis of Individual Differences in Multidimensional Scaling via an N-way Generalization of 'Eckart-Young' Decomposition." *Psychometrika*, 35:283-320.